CHAPTER 2 NOTES AND CLASS MATERIAL

These notes/worksheets are meant to *briefly* summarize the material covered in Chapter 2 of our text, section by section, and give you problems to work out yourself in class. For fuller explanations and relevant examples, you should read the textbook. My hope is that these notes will be useful when you are *reviewing* what you have already learned, but they probably won't be good enough all by themselves to learn the material from scratch.

1. Rectangular Coordinate Systems

The two essential facts you need to take away from this section are the two following algebraic formulas which give geometric information about points in the plane:

The Distance Formula: Given a pair of points (x_0, y_0) and (x_1, y_1) in the plane, the distance between them is equal to $\sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}$.

The Midpoint Formula: Given a pair of points (x_0, y_0) and (x_1, y_1) in the plane, the midpoint of the line segment connecting them has coordinates $\left(\frac{x_0+x_1}{2}, \frac{y_0+y_1}{2}\right)$.

To really *know* these formulas, you first must know *why* they are true, so that you can never forget them. Then you must be able to *use* them as tools for solving problems.

Why the distance formula is true: Assuming the line segment connecting the two points (x_0, y_0) and (x_1, y_1) isn't horizontal or vertical, we can draw a right triangle whose hypotenuse is the line segment connecting our two points, as in Figure 1. The other two sides of the triangle have lengths $|x_0 - x_1|$ and $|y_0 - y_1|$, and so by the Pythagorean Theorem the length of the hypotenuse h of this triangle (which is also the distance between our two points), satisfies the equation $h^2 = |x_0 - x_1|^2 + |y_0 - y_1|^2$. Taking the square root of each side of this equation and removing the absolute values, we get out

$$h = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}.$$

Check your understanding: Why do the smaller sides of the right triangle in Figure 1 have lengths $|x_0 - x_1|$ and $|y_0 - y_1|$? Why were we able to remove the absolute value signs from the distance formula in the last step? Is the distance formula still true when both points lie on a horizontal or vertical line?

Basic Use of the Distance Formula: Find the distance between the following pairs of points.

(1) $(2, -1)$ and $(3, 1)$.

(2) $(-1, -3)$ and $(-4, 2)$.

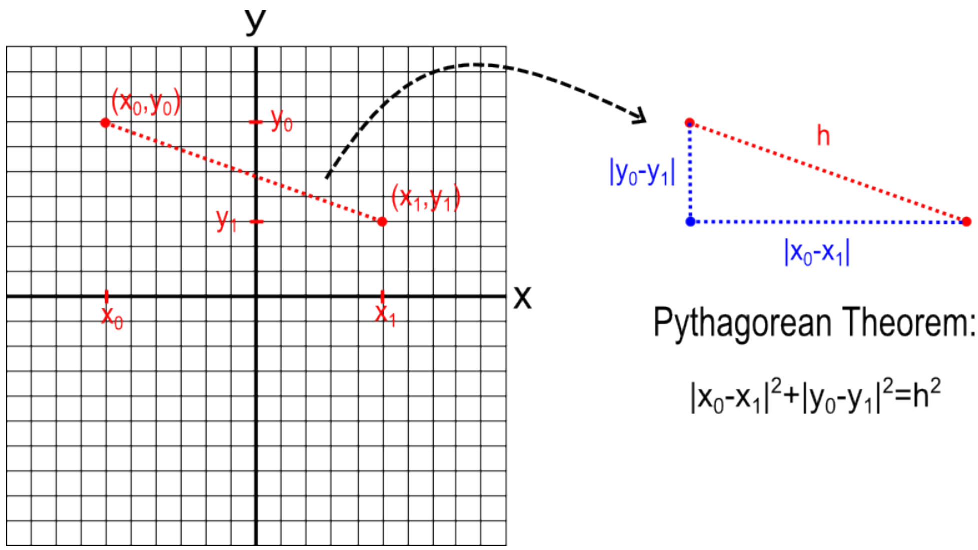

FIGURE 1. Why the distance formula is true

It is important to be aware that the distance formula is not just a tool for finding the distance between a given pair of points. This is just its most basic use. More generally, we think of it as a tool which allows us to translate geometric questions about distance in the plane into algebraic questions.

Problem: Find an algebraic formula which describes the set of all points which lie on the *perpendicular bisector* of $(1, 2)$ and $(4, 0)$. (Hint: One way of describing the perpendicular bisector of a pair of points is as the set of all points in the plane which are equidistant from that pair).

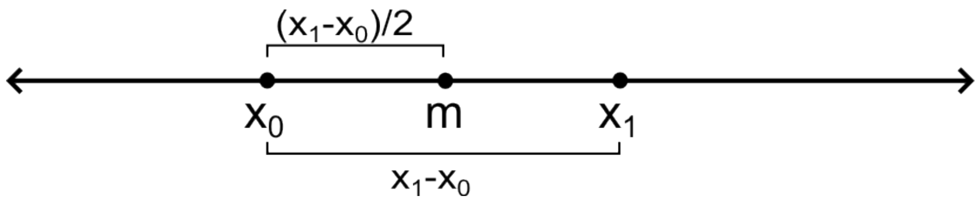

FIGURE 2. The midpoint formula on the line

Problem: Find the set of points on the horizontal line $y = 1$ which are a distance of 3 units away from the point $(0, 2)$.

Why the midpoint formula is true. We'll start by finding the midpoint of two points x_0 and x_1 on the line. Assuming we have labeled them so that $x_0 < x_1$, the midpoint between x_0 and x_1 on the number line is the point $x_0 + \frac{1}{2}(x_1 - x_0)$, as can be seen from Figure 2, which reduces algebraically to $\frac{x_0+x_1}{2}$. In other words, the midpoint between x_0 and x_1 is just the average of x_0 and x_1 on the number line.

The midpoint of a pair of points (x_0, y_0) and (x_1, y_1) in the plane, is found similarly. You just take the average value of the x values and the average value of the y values separately.

Problem: Find the midpoints of the following pairs:

(1) $(1, 2)$ and $(-1, -2)$

(2) $(-3, 1)$ and $(2, 7)$

2. Graphs of Equations

The main point of this section is just to remind you of how the graph of an equation is defined. However, to illustrate the concept (and to prepare you for future work) we will be taking a close look at the graphs of circles in particular. Working with circles will be the meat of this section, as far as problem solving is concerned.

A *solution* to an equation with one variable is just any number which makes the equation come out true when it is plugged in for the variable.

For example, consider the equation $x^2 + 1 = 2x$, if you plug in $x = 1$ you get out $1^2 + 1 = 2 \cdot 1$, which is true, whereas if you plug in $x = 0$ you get $0^2 + 1 = 2 \cdot 0$, which is false. Therefore $x = 1$ is a solution but $x = 0$ is not, and in fact it is possible to show that $x = 1$ is the *only* solution to the original equation. Usually an equation involving a single variable only has a finite number of solutions, if any.

A *solution* to an equation with *two* variables (we'll use x and y) is just a pair of numbers (a, b) which makes the equation come out true when a is plugged in for x and b is plugged in for y.

For example, consider the equation $x^2 + y = 1$. Then $(0, 1)$ is a solution because we get $0^2 + 1 = 1$, and so is $(2, -3)$ because $2^2 + (-3) = 1$, whereas $(2, 0)$ is not because the resulting equation $2^2 + 0 = 1$ is false.

The *graph* of an equation in two variables is just what you get by plotting out all of its solutions in the Cartesian plane. So for example the graph of $x^2 + y = 1$ is an upside down parabola. It is possible for the graph of an equation to fill up the entire plane, or to consist of no points at all. For most equations, however, it will be a curve.

x-intercepts and y-intercepts. An *x-intercept* of a curve is any point on the curve which touches the x-axis. The x-axis is the set of all points in the plane of the form $(x, 0)$, so if a curve is given algebraically by an equation $f(x, y) = g(x, y)$, its x-intercepts can be found by plugging in 0 for y and solving for x. Similarly, we can find the y-intercepts by setting $x = 0$ and solving for y.

Problem: Find the x and y intercepts (if any) of the graph of the equation

$$x^3y^2 - 2xy^2 - 5 = y + 2x^2 + x.$$

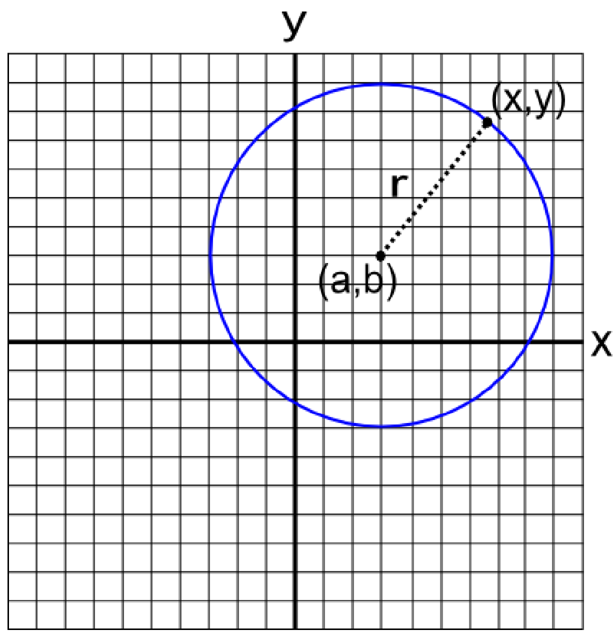

FIGURE 3. Circle of radius r centered at (a, b).

The Graph of a Circle. Given *any* circle with fixed center (a, b) and radius r in the plane, pictured in Figure 3, it is easy to find an equation which defines it. By the distance formula, a point (x, y) is r units away from (a, b) if and only if $r = \sqrt{(x-a)^2 + (y-b)^2}$, so the set of all points (x, y) which solve this equation is precisely the circle we are after. Squaring both sides then gives us the standard form.

The standard equation of a circle. *The circle of radius r centered at (a, b) is defined by the equation*

$$r^2 = (x-a)^2 + (y-b)^2.$$

Basic use of the standard equation: Find the standard equation of the circle of radius 3 centered at $(4, -2)$. Then find the x and y intercepts of the circle, if any.

More advanced use of the standard equation: Let $\mathcal{C}$ be the circle of radius 4 in the second quadrant which is tangent to the x and y axes. Find the points (if any) where this circle intersects the line $x = -2$.

In the previous two problems, we started with a geometric description of the circle and used the circle formula to transform this into an equation. However, for equations of the right form, it is possible to use the method of "completing the square" to get the standard form of the equation of a circle.

Problem: Find the center and radius of the circle defined by the equation $x^2 - 6x + y^2 + 4y + 3 = 0$.

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